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The fluid mechanics of the ureter from a lubrication theory point of view

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The function of a healthy ureter is analyzed in terms of a fluid-mechanical model. To the extent that the Reynolds number is of the order of one, the fundamental equations are shown to reduce to those of the theory of lubrication. It is found that from the point of view of the pressure variation with time (the urometrogram) the important part of the peristaltic wave is the constricting part. For this reason this part of the wave is represented with an algebraic expression of the form $h \sim x^n$ making it possible to find closed form solutions. Using Fourier analysis in defining the complete wave shape of the ureter it was also possible to obtain numerical solutions. For both cases it is shown that there is good agreement between the theoretical and experimental pressure distributions, this not being the case for sinusoidal wave shapes. An approximate equation for the flux is developed and a universal relation is presented connecting the maximum pressure, flux and kinematic behaviour of the ureter.

1. Introduction

In 1966 Latham investigated the fluid mechanics of peristaltic pumps and since then, other papers on the same subject have followed by Burns & Parkes (1967), Hanin (1968), Barton & Raynor (1968), Shapiro, Jaffrin & Weinberg (1969), and Yin & Fung (1969), among others. These papers are useful contributions to the understanding of peristaltic pumping, but their relevance to the problem of the ureter is not investigated by the authors. In fact, the pressure distribution inside such a pump, which is important information when one examines the ureter, is not dealt with and furthermore, when this is done, it can be shown that the ureter does not function as a sinusoidal peristaltic pump. This was pointed out by Lykoudis (1966) where the emphasis was not in developing a peristaltic pump from an engineering point of view (where, for instance, its mechanical efficiency would be important) but rather in trying to understand what peristaltic motion meant to the ureter. For this reason the urometrogram, which is the hydraulic signature of the ureter, became the object of interest since this has been the primary source of hydraulic information obtained by all urologists over many years. In this paper† an attempt will be made to discuss the

† The present paper should be considered a companion of the work by Lykoudis (1969) in which the presentation is made having in mind the urologist and where the mathematics presented here, are absent.

function of the ureter using the urometrogram as the fundamental criterion against which all fluid mechanical theories will be tested.† In building up such theories the following physiological data related to the function of a healthy ureter will be accepted: (*a*) Timewise, at any given point inside the ureter the pressure remains constant at the level of about 4 mm Hg (this is the so-called resting pressure) and then suddenly rises to a peak of the order of 30 mm Hg to fall abruptly again to the level of the resting pressure. (*b*) The resting pressure at *all* points inside the ureter is the same. There is no average pressure gradient favourable or adverse that exists in the healthy ureter. (The case of reflux for which an adverse pressure gradient might exist, as is explained in Bergman (1967), will not be considered here.) (*c*) The peristaltic motion of the ureter does not produce pressures lower than the resting pressure. (*d*) Inasmuch as one can judge from radiological evidence, the collapsed part of the ureter appears to be totally occluded.

It will be shown here that because of the nature of these data the fundamental fluid-mechanical equations reduce to those of the classical theory of lubrication. The time dependence of the problem does not enter through the non-steady inertial term (since the inertia forces are found to be very small and hence negligible) but only through the time dependence ascribed to the moving boundary. The results of the theoretical analysis will be applied to the actual kinematic and geometric data of the human ureter and it will be seen that they are compatible with observations. It will also be shown that sinusoidal pumps of the type described by Latham (1966) and Shapiro *et al.* (1969) are not capable of describing the actual function of the ureter.

2. Governing equations and numerical solution

First, we recognize that the working medium (urine) is incompressible and thus behaves dynamically as water does. The forces present in any fluid-mechanical system are the forces due to inertia, pressure, and viscosity which at any given time must be in balance.‡ A great simplification occurs if we make an estimate of the Reynolds number.

For the ureter the Reynolds number is given by the following expression

$$R_e = ca^2/\nu\lambda, \tag{1}$$

where c is the wave speed, a the characteristic length in radial direction, ν the kinematic viscosity, and λ the wavelength (see figure 1). Introduction of some typical values pertaining to the ureter, $c = 3\,\text{cm/sec}$, $a = 1{\cdot}5\,\text{mm}$, $\lambda = 15\,\text{cm}$ and $\nu = 0{\cdot}007\,\text{cm}^2/\text{sec}$ (this is the viscosity of water at 40 °C, since urine is essentially water) gives $R_e = 0{\cdot}65$. This is an average value, while in extreme cases the Reynolds number might go as high as 1·5. Because of this, one can feel somewhat confident in neglecting the inertia forces, so that the forces due to pressure must balance those due to viscosity. Assuming also that $\lambda \gg a$ we can write the

† The interested reader will find more information describing the physiology and function of the ureter in the works by Kiil (1957) and Bergman (1967).

‡ The role played by the gravitational forces is examined by Lykoudis (1969).

equations of conservation of momentum in an axisymmetric co-ordinate system moving with the wave shape at speed c as follows:

$$\frac{\partial p}{\partial x} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2}$$

and

$$\frac{\partial p}{\partial r} = 0 \quad \text{or} \quad p = p(x). \tag{3}$$

The equation of conservation of mass is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r} = 0. \tag{4}$$

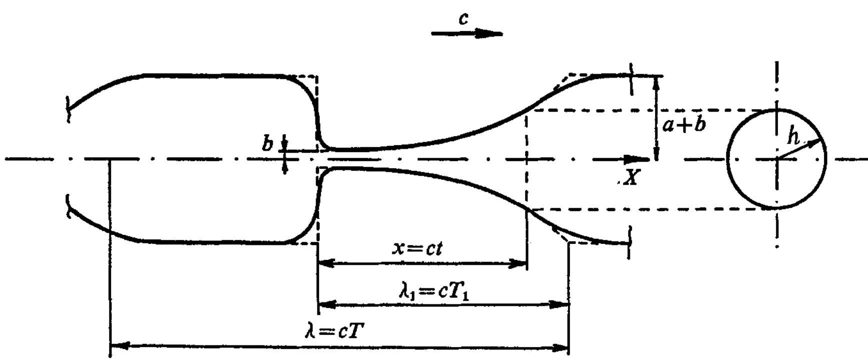

FIGURE 1. Co-ordinate system and geometry.

The boundary conditions in this moving co-ordinate system are:

$$u = -c \quad \text{at} \quad r = h, \tag{5}$$

$$\partial u/\partial r = 0 \quad \text{at} \quad r = 0, \tag{6}$$

$$v = 0 \quad \text{at} \quad r = 0. \tag{7}$$

Since the pressure is only a function of x, (2) can be directly integrated twice using the boundary conditions. We obtain

$$u = -c-\frac{1}{4\mu}\frac{dp}{dx}(h^2-r^2). \tag{8}$$

Equation (8) gives the axial velocity of the fluid in the moving system in terms of the local pressure gradient. As in the theory of lubrication (Schlichting 1960) this pressure gradient has to be found from an expression for the volume flow which is constant in this moving co-ordinate system. So

$$q = 2\pi\int_0^h ur\,dr. \tag{9}$$

Substitution of (8) and integration gives

$$\frac{dp}{dx} = -\frac{8\mu}{\pi h^4}(q+\pi ch^2). \tag{10}$$

The radial velocity v is found by integrating (4) with respect to r, after substitution of $\partial u/\partial x$. This last one is found from (8) with (10) inserted. Equation (10) expresses the pressure gradient dp/dx in terms of the constant, however unknown, flux q. In order to find q we have to set a condition for the pressure distribution along the tube. This pressure distribution is given by

$$p_x = p_0 + \int_0^x \frac{dp}{dx}\,dx. \tag{11}$$

Here p_0 is the pressure at $x = 0$. Guided by the information from urometrograms we assume that the pressure gradient over a wavelength is zero or

$$\Delta p_\lambda = p_\lambda - p_0 = \int_0^\lambda \frac{dp}{dx}\,dx = 0. \tag{12}$$

Substitution of (10) in (12) gives the following expression for q

$$q = -\pi c \frac{\int_0^\lambda h^{-2}\,dx}{\int_0^\lambda h^{-4}\,dx}. \tag{13}$$

Up to this point, it was not necessary to specify the shape of the wall, h. In the work by others, mentioned before, h was always taken to be sinusoidal which made it possible to obtain algebraic solutions for the integrals involved.

A comment must now be made with respect to the assumed radial motion of the boundary of the ureter. X-ray urography shows that for normal ureters the portion of the length which corresponds to the contracted phase is longer than the one corresponding to the relaxed phase. Also, as will be shown later, the assumption of h being sinusoidal does not produce a urometrogram as they are observed. It was therefore decided to postulate a wave shape more compatible with the observations and to judge it through the obtained pressure distribution.

It is obvious that in the case of an arbitrary wave shape, an algebraic solution of the integrals is impossible, but it is possible to obtain urometrograms through numerical integration by a computer. In order to do so we have defined the geometry and the timewise behaviour of the ureter through Fourier analysis. The pressure distribution relative to the resting pressure has then been obtained from (11) after substitution of (10) and (13). The numerical results are compared with two examples taken from Kiil (1957) for which information about the wave speed is available. The results are given in figure 2. Figure 2(*a*) shows the shape of the ureter on a more or less real scale, while figure 2(*b*) shows the same with the radial scale highly exaggerated for the purpose of illustration. The theoretically and experimentally obtained urometrograms are given in figure 2(*c*).

Several numerical integrations of the type mentioned above have been produced and we have come to the following conclusions. The pressure is very much different from the resting pressure (equal to zero in figure 2) only during the contraction phase when the lumen is very narrow. At this point, we recall that in all lubrication theories the pressure rises abruptly only when the lubrication film is very thin.

It follows from (10) that for a general wave shape the pressure will exhibit two extrema, located at those two places where the axial velocity (measured in a stationary frame of reference) is zero. In this context the maximum pressure occurs during the collapsing phase (and before the minimal diameter is reached) while the minimum pressure occurs at the relaxation phase of the ureter and at a point which will have the same diameter as the diameter corresponding to the maximum pressure.† In the case of the ureter this means that the pressure will dip below the resting pressure. However, this is normally not observed in urometrograms, which leads us to believe that such negative pressures are too small to be identified. On the other hand, from a theoretical point of view one can always choose the wave shape in such a fashion as to make this negative pressure as small as desired. This was done for instance in figure 2 where a small negative pressure can be seen.

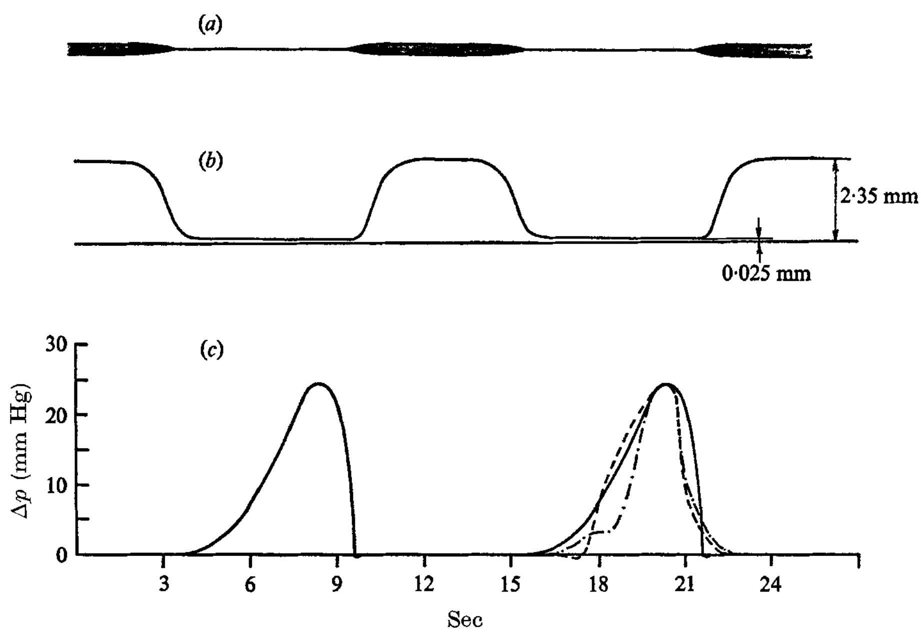

FIGURE 2. Experimental and theoretical urometrograms as obtained from numerical calculations. —, theory (numerical solution); – – –, —·—, experiment (Kiil 1957, p. 59).

From the above we learn that as far as the urometrogram is concerned the exact kinematic behaviour of the ureter is important only when the motion is in the almost fully constricted phase.‡ On the other hand, because of the fact that on one side of the wave the lumen is practically fully occluded, for flux calculations, only the volume of the urine pool is significant.

† The presence of the maximum and minimum in the pressure distribution is also found for instance in the theory of rolling bearings; in this case the negative pressures are disregarded or circumvented by the use of appropriate boundary conditions as explained in Cameron (1966).

‡ This was also pointed out by Lykoudis (1966, 1969).

Because of these considerations and in order to obtain a better physical understanding it was decided to look for a closed form solution of the problem, using some of the above findings.

3. Lubrication theory solution

Let us assume a simple relation of the form $h \sim x^n$ for the constriction phase where n is an arbitrary constant. In such a case, the only condition implied on n will be that it should be larger than or equal to one since for $n < 1$ the curvature during the final stages of collapse is not compatible with physiological observation. The higher value for n, the flatter the ureter at the point of its smaller radius. This choice of $h \sim x^n$ is guided by two considerations, the first being that Lykoudis (1966) found that if one assumes a stationary collapsing tube in which the time behaviour of the boundary shortly before its total collapse to the minimum radius was sinusoidal,† this sinusoidal hypothesis could be approximated by $h \sim x^2$. Secondly, and this is shown in the footnote following (30), for a sinusoidal peristaltic pump, the assumption of $h \sim x^2$ is also a good approximation in obtaining the maximum pressure, which occurs shortly before the minimal diameter is reached.

Before entering into the details of computation, let us apply an order of magnitude argument in an effort to derive a dimensionally correct result for the maximum pressure.

Imagine that *during its contraction* the ureter collapses with a velocity V while at the same time the wave shape travels in the axial direction with the wave speed c. In order to fix further the ideas, consider the geometry of figure 1. We see there that we have two characteristic dimensions in the Y direction, the minimum radius b and the maximum radius R which is equal to $(a+b)$. In this case a is the amplitude of the peristaltic wave. Since in the case of the ureter, b is much smaller than R, the amount of fluid displaced in the Y direction with the velocity of the boundary V will go out through a cross-sectional area corresponding to the radius h, at a station x, rather than b. An order of magnitude argument applied to the equation of conservation of mass yields

$$U \sim 2Vx/h. \tag{14}$$

On the other hand, the viscous forces per unit volume acting in the direction X will be based on the rate of change of the velocity U within the radius b so that the equation for the conservation of momentum in the direction y yields

$$\Delta p \sim \frac{\mu U x}{b^2} \sim \frac{\mu x}{b^2}\left(2\frac{Vx}{h}\right) \sim 2\mu\frac{Vx^2}{hb^2}. \tag{15}$$

We now assume that the geometry of the boundary during its collapse is given by the relation

$$h = b + a(t/T_1)^n. \tag{16}$$

In the above, T_1 is the time in which h changes from b to $a+b$ or $T_1 = \lambda_1/c$. We set

$$V = dh/dt = \frac{nat^{n-1}}{T_1^n} \tag{17}$$

and note that $x = ct$.

† In contrast to this hypothesis being valid for *all* of the ureter.

Hence, (15) becomes
$$\Delta p \sim \frac{\mu c^2}{b^2} \frac{t^{n+1}}{[(b/a)T_1^n + t^n]}. \tag{18}$$

We are now interested in obtaining the maximum of Δp timewise. This will occur where $d(\Delta p)/dt = 0$. After some algebra, we find the following relations (constants apart):
$$\Delta p_{\max} \sim \frac{\mu}{T_1}\left(\frac{\lambda_1}{a}\right)^2 \left(\frac{a}{b}\right)^{2-1/n}, \tag{19}$$

or
$$\Delta p_{\max} \sim \frac{\mu c \lambda_1}{a^{1/n} b^{2-1/n}}. \tag{20}$$

The exact analysis which follows will only determine the constant of proportionality. The first conclusion from this formula is that since λ_1, T_1 and a are fixed, the maximum pressure is inversely proportional to a power of the minimum radius b which at most could be the second (corresponds to $n \to \infty$), and at least would be equal to one (corresponding to $n = 1$).

Let us now proceed with the details of the problem. In the moving co-ordinate system h is defined by
$$h = b + a(x/\lambda_1)^n. \tag{21}$$

The condition on p_x given by (12) is changed into
$$\Delta p_{\lambda_1} = p_{\lambda_1} - p_0 = \int_0^{\lambda_1} \frac{dp}{dx}\, dx = 0. \tag{22}$$

As a result of this λ in (13) is changed into λ_1 and after substitution of (21), integration gives
$$q = -\frac{6n^2}{(3n-1)(2n-1)} \pi c b^2, \tag{23}$$

while we have assumed that $a + b \gg b$. With (23) the expression for the local pressure gradient becomes
$$\frac{dp}{dx} = -\frac{8\mu c}{h^4}\left[h^2 - \frac{6n^2}{(3n-1)(2n-1)} b^2\right]. \tag{24}$$

Insertion in (11) gives us the pressure distribution
$$p_x = p_0 - 8\mu c \int_0^x \frac{dx}{h^2} + 8\mu c b^2 \left(\frac{6n^2}{(3n-1)(2n-1)}\right) \int_0^x \frac{dx}{h^4}, \tag{25}$$

where h is given by (21). Through integration we obtain an analytic expression for the pressure distribution
$$p_x = p_0 + 8\mu c b^2 \left(\frac{6n^2}{(3n-1)(2n-1)}\right) \left[\frac{x}{3nb[b + a(x/\lambda_1)^n]^3} + \frac{(3n-1)x}{6n^2 b^2 [b + a(x/\lambda_1)^n]^2}\right]. \tag{26}$$

Figure 3 shows the shape of the boundary and the pressure distribution relative to the resting pressure, $\Delta p_x = p_x - p_0$, calculated for a range of values of n and one value for a, b, λ_1 and c. We see that $\Delta p_{\max}$ and the length over which Δp_x is larger than zero are highly dependent on n. $\Delta p_{\max}$ can be found in the following way. Δp is maximal at the location $x = x_{\max}$ for which $dp/dx = 0$. This condition gives
$$h_{\max}^2 = -\frac{q}{\pi c} = \frac{6n^2}{(3n-1)(2n-1)} b^2, \tag{27}$$

which gives us the radius $h_{\max}$ at $x = x_{\max}$. But since we know $h_{\max}$ from (21) we can find an expression for $x_{\max}$

$$x_{\max} = \lambda_1\left(\frac{b}{a}\right)^{1/n}\left[\left(\frac{6n^2}{(3n-1)(2n-1)}\right)^{\frac{1}{2}} - 1\right]^{1/n}. \tag{28}$$

Substitution of (27) and (28) into the pressure distribution (26) makes it possible to give an expression for the maximum pressure (relative to the resting pressure)

$$\Delta p_{\max} = p_{\max} - p_0 = g(n)\,\frac{\mu c \lambda_1}{a^{1/n} b^{2-1/n}}, \tag{29}$$

where $$g(n) = \frac{4}{3n}\left[\frac{6n^2}{(3n-1)(2n-1)} - 1\right]^{1/n}\left[\left(\frac{(3n-1)(2n-1)}{6n^2}\right)^{\frac{1}{2}} + \frac{(3n-1)}{2n}\right]. \tag{30}$$

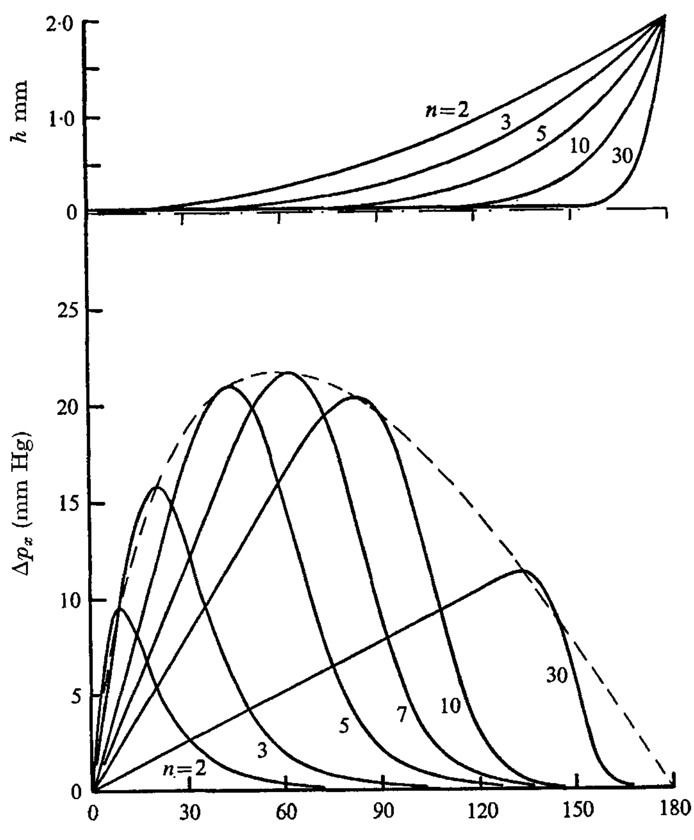

FIGURE 3. The geometry and pressure distribution as a function of the exponent n according to lubrication theory. $a = 2{\cdot}0$ mm, $b/a = 0{\cdot}01$, $c = 30$ mm/sec, $\lambda_1 = 180$ mm, $h = b + a\,(x/\lambda_1)^n$.

We see that (29) is essentially the same as (20). The rigorous analysis provided only the constant $g(n)$ which is important in case of numerical calculations, but does not add to the understanding of the physics of the problem.†

† In the case of $n = 2$, $g(n) = 1{\cdot}05$. An exact analysis for a sinusoidal wave shows that in the notation of the present work ($\lambda = 2\lambda_1$) the maximum pressure is given by

$$\Delta p_{\max} = 0{\cdot}9\,\frac{\mu c \lambda_1}{a^{\frac{1}{2}} b^{\frac{3}{2}}}.$$

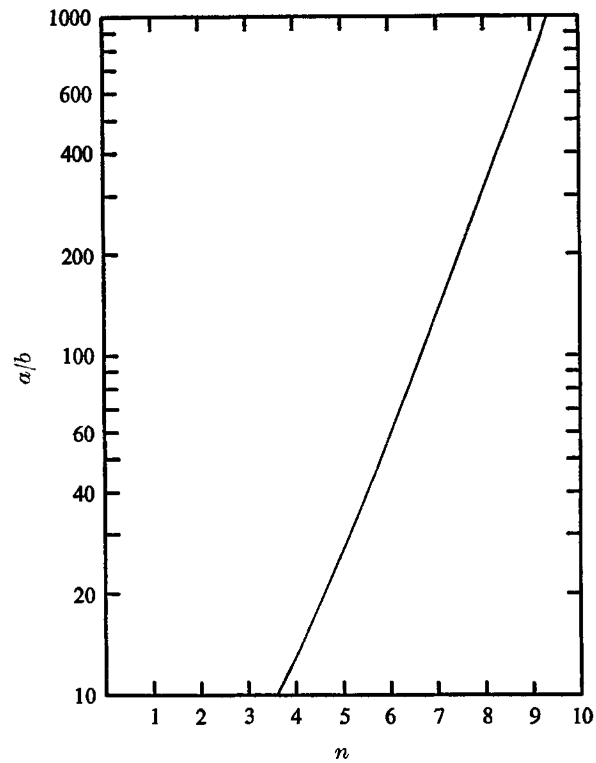

FIGURE 4. Solution of transcendental equation which determines for every a/b the power n for which Δp_{max} is maximal.

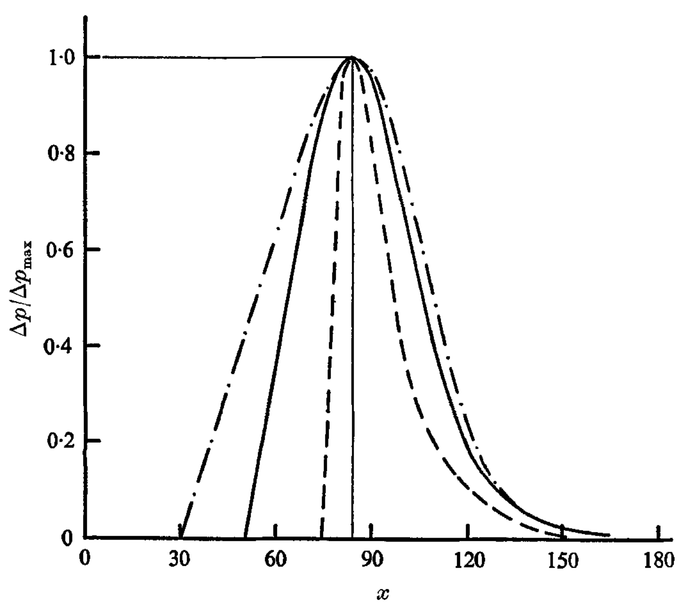

FIGURE 5. The normalized pressure distribution $\Delta p/\Delta p_{max}$ as a function of n for a fixed b/a. The maxima are located at the same x. $a = 2$ mm, $b/a = 0{\cdot}01$, $c = 30$ mm/sec, $\lambda_1 = 180$ mm. - - -, $n = 2$; —, $n = 4$; —·—, $n = 6$.

In order to find the maximum of $\Delta p_{\max}$ we have to differentiate (29) with respect to n. This gives a complicated transcendental equation which was solved on the computer. The solution is given in figure 4, where for every value of a/b one can find the n for which $\Delta p_{\max} = (\Delta p_{\max})_{\max}$.

In order to investigate the influence of n and the geometry on the shape of the pressure curves, aside from a change in the maximum pressure, these pressure distributions were normalized with their $\Delta p_{\max}$ and their maxima were put at the same location. From (26) and (29) one can find that $\Delta p/\Delta p_{\max}$ is a function of

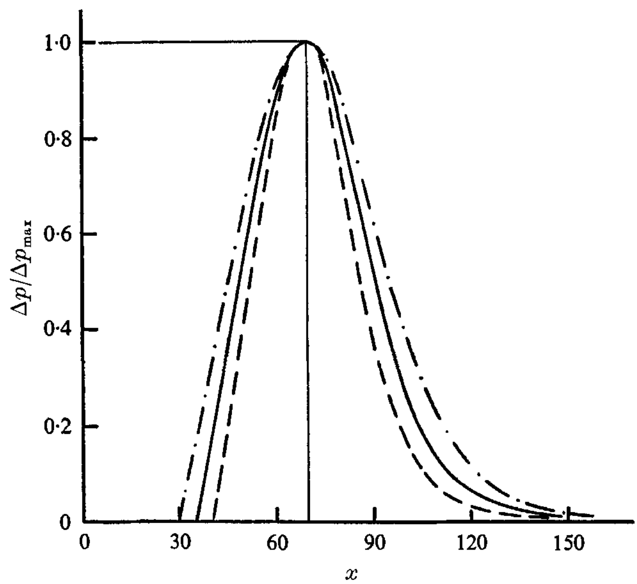

FIGURE 6. The normalized pressure distribution $\Delta p/\Delta p_{\max}$ as a function of b/a for a fixed n. The maxima are located at the same x. $a = 2$ mm, $c = 30$ mm/sec, $\lambda_1 = 180$ mm, $n = 4$. —·—, $b/a = 0{\cdot}02$; —, $b/a = 0{\cdot}01$; – – –, $b/a = 0{\cdot}005$.

n, b/a and x/λ_1 only. Figure 5 shows the normalized pressure as a function of n for the same values of a, b and λ_1 as used in figure 3. It can be seen that the width of the pressure pulse increases distinctly with n, while the slope on both sides becomes less steep. Figure 6 gives $\Delta p/\Delta p_{\max}$ as a function of b/a for $n = 4$ and again the same values for a and λ_1. Here one can observe that the width decreases and the slope steepens with diminishing b/a. It follows from these figures that the change in $\Delta p_{\max}$ is much more pronounced with a change in b/a than in n. It is therefore concluded that in general, the order of magnitude of $\Delta p_{\max}$ is determined by b/a and the shape of the curve by n.

4. Comparison of the theory with experimental observation

To test this simple solution with the observations by the urologist we again turn to Kiil (1957) for some data. Figure 7 shows an experimental urometrogram with one obtained from the theory as a function of time. The conversion from x

in the moving co-ordinate system to t in a fixed system is done by realizing that $t = -(x+z)/c$ (where z is the co-ordinate along the tube axis in a fixed system). After λ_1, a and c are fixed compatible with X-ray observations, it is found that b/a has to be of the order of $1/100$ to obtain $\Delta p_{\max}$ and that in this range the shape requires n to be equal to 4. We see that both urometrograms coincide fairly well except for the tail. However, we believe this part of the pressure distribution to be connected with the initial expansion of the ureter after its collapse. It is believed that in this region, which is not covered by the theory, the presence of the catheter

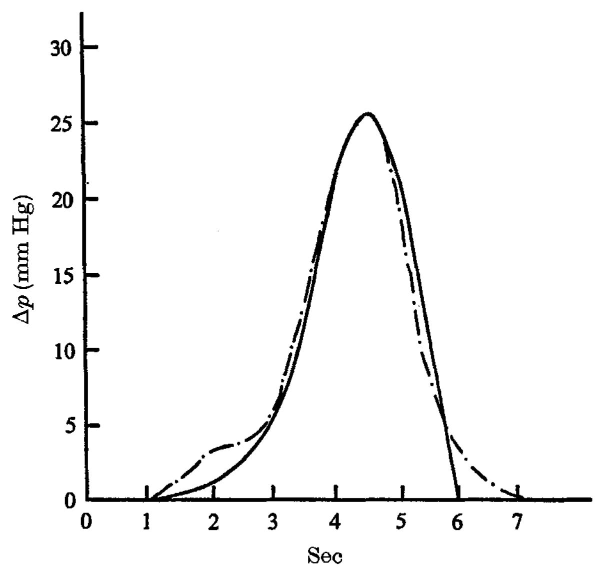

FIGURE 7. Experimental and theoretical urometrograms as obtained from lubrication theory. $a = 2{\cdot}35$ mm, $b = 0{\cdot}02$ mm, $c = 30$ mm/sec, $\lambda_1 = 250$ mm, $n = 4$. —, present theory; —·—, experiment (Kiil 1957, p. 59).

becomes important. Figure 8 shows the corresponding theoretical axial velocity distribution. Comparison with experimental results is impossible since measurements of this type have not been made.

In order to see what a peristaltic pump of the type described by Shapiro *et al.* (1969) produces as a urometrogram, their theory has been used to compute the pressure variation with time for the same wavelength and geometry found compatible with the real urometrogram of figure 2. It is seen in figure 9 that apart from the negative pressures, the peak pressure $\Delta p_{\max}$ is much smaller for the sinusoidal ureter. It is also seen there that the time over which the pressure stands much above the resting pressure is much smaller than the one needed to correlate with the actual urometrogram. This is adequate proof that the ureter does not behave as a sinusoidal wave.

From the example cited it is seen that a fairly large contraction ratio is needed in order to obtain the peak pressures recorded in urometrograms. This ratio (b/a) seems of the order of $1/100$. This, of course, is impossible to measure, not only because of the small diameters involved, but also because of the fact that the ureter in its collapsed state has a starfish-shaped cross-section. At first sight this is an alarming result since the size of the catheters used for the measurement of pressures inside ureters is more than 1 mm in diameter whereas the theory

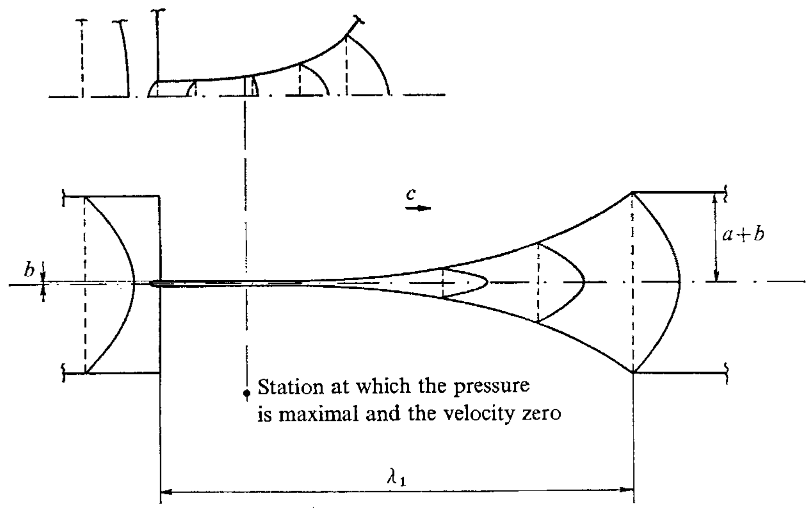

FIGURE 8. Theoretical axial velocity distribution as obtained from lubrication theory. $a = 2{\cdot}35$ mm, $b = 0{\cdot}02$ mm, $c = 30$ mm/sec, $\lambda_1 = 250$ mm, $n = 4$.

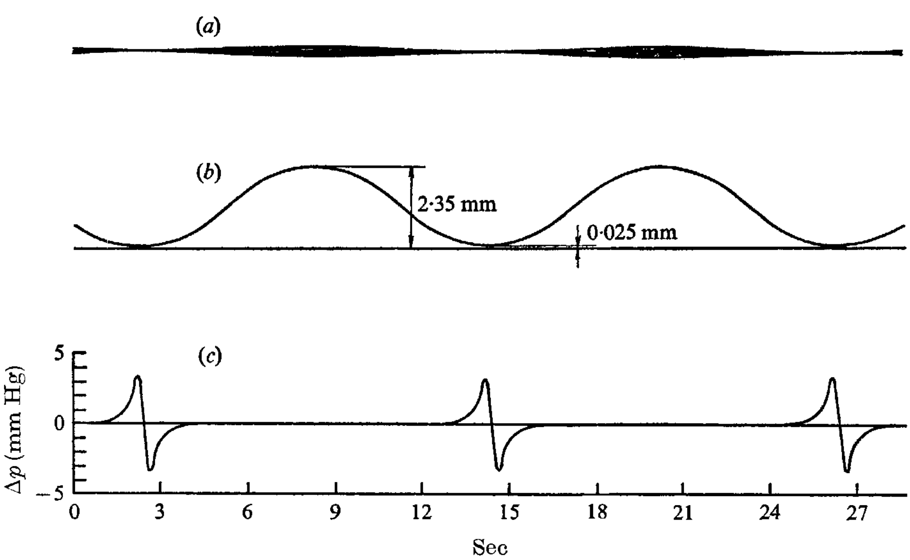

FIGURE 9. Theoretical urometrogram according to the sinusoidal theory.

dictates minimum lumens of the order of 0·05 mm! The correct interpretation of this diameter is that there is a film of urine around the catheter which behaves like a thin layer of oil† capable of sustaining high pressures as indeed we observe in

† See, for example, Prandtl (1954, pp. 158) where an oil film of 0·0333 mm can sustain a pressure of 23,000 mm Hg! Such films are difficult to measure, but their existence is postulated successfully in design.

load carrying oil bearings. The fundamental theory of the ureter is a direct example of such lubrication theories.

A further check of the findings of this work can be made in terms of recently obtained experimental information on the time elapsed between the beginning of the occlusion of the ureter and the time at which the maximum pressure appears. For dogs these data have been obtained by Barry (1969) and seem to indicate that this time interval is of the order of 1·6 seconds. For the two cases which were taken from Kiil (1957) and involve humans our theory gives 5 seconds. Quantitative comparison of these results, however, should not be attempted, since for the experiment no data are given regarding wave speed, contraction length and time between two contractions. More important is that a time interval was found indeed even before it was experimentally verified, since some urologists have thought it to be an artifact† introduced by the measurement itself. It seems that our analysis leaves little doubt regarding its reality and explanation in terms of the present theory. In fact, it is a feature present in all lubrication film theories.

5. Functional relation between maximum pressure and flux

In order to evaluate further some of the theoretical findings presented in this work, we need to recast some of the previously found results into quantities that are more directly amenable to observations, especially of the type which a urologist measures. One of these observations is the urine flux, and so we have to evaluate a theoretical expression for this.

In a stationary co-ordinate system the flux Q at a certain cross-section is given by

$$Q = q + \pi c h^2. \tag{31}$$

Averaged over a total wavelength this becomes

$$\bar{Q} = q + \frac{\pi c}{\lambda}\int_0^{\lambda_1} h^2 dx + \frac{\pi c}{\lambda}\int_{\lambda_1}^{\lambda} (a+b)^2 dx. \tag{32}$$

Evaluation of the integrals after insertion of (21) and (23) gives

$$\bar{Q} = \pi c b^2\left[1 - \frac{6n^2}{(3n-1)(2n-1)}\right] + \frac{\pi c}{\lambda}\left[\frac{2ab\lambda_1}{(n+1)} + 2ab\lambda_2\right] + \frac{\pi c a^2}{\lambda}\left[\lambda_2 + \frac{\lambda_1}{2n+1}\right], \tag{33}$$

where $$\lambda_2 = \lambda - \lambda_1.$$

Since $b \ll a$ we make the following approximation:

$$\bar{Q} \cong \pi c a^2\left[\frac{\lambda_2}{\lambda} + \frac{\lambda_1}{\lambda}\left(\frac{1}{2n+1}\right)\right], \tag{34}$$

or $$\bar{Q} \sim \pi c a^2. \tag{35}$$

Using (35) we can write (29) in the following form

$$\Delta p_{\max} \sim \mu\lambda c^{1+1/2n}/b^{2-1/n}\,\bar{Q}^{1/2n}, \tag{36}$$

or $$\Delta p_{\max} \sim \mu T\,(c^{2+1/2n}/b^{2-1/n})\,(1/\bar{Q}^{1/2n}). \tag{37}$$

† See the discussion in *Hydrodynamics of the Ureter* (Proceedings of the Workshop on the Hydrodynamics of the Upper Urinary Tract).

These equations tell us that for any normal ureter there is a relation that binds together five quantities, $\Delta p_{\max}$, b, C, Q, and λ or T. This observation is very instructive because it indicates that any effort to correlate only two of the above quantities when the others are not controlled and fixed is impossible. This then explains why observers have found it possible to measure an increased or decreased frequency of contraction with increased Q. For instance, in dealing with this problem Kiil (1957, p. 70) makes the following statement: "The frequency of ureteral contractions gives no information about the total urine flow.... The frequency of ureteral contractions did not give any information about the urine flow issuing from each ureter. The ureter with the highest rate of contractions might transport the least volume of urine. The frequencies of ureteral contractions might be very different at equal flows.... Most of the data revealed a trend towards more frequent ureteral contractions when the urine flow increased.... In almost no case were the changes in urine flow and the changes in the rate of ureteral contractions directly proportional."

Unfortunately, there are not adequate data† known to the authors which could serve for the determination of the constant of proportionality and the free parameter n of (36) and (37). If we had such information, we could perhaps derive a *universal relation* for the ureter which would link all of its fluid-mechanical parameters, notwithstanding the range over which they may take from instance to instance.

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† An attempt by Lykoudis (1966) to use such data from Kiil's book cannot be considered as conclusive.